CONNECTEDNESS OF KISIN VARIETIES ASSOCIATED TO ABSOLUTELY IRREDUCIBLE GALOIS REPRESENTATIONS

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Abstract. We consider the Kisin variety associated to a n-dimensional absolutely irreducible mod p Galois representation $\bar{\rho}$ of a p-adic field K togther with a cocharacter μ. Kisin conjectured that the Kisin variety is connected in this case. We show that Kisin's conjecture holds if K is totally ramified with $n = 3$ or μ is of a very particular form. As an application, we get a connectedness result for the deformation ring associated to $\bar{\rho}$ of given Hodge-Tate weights. We also give counterexamples to show Kisin's conjecture does not hold in general.

1. Introduction

Let K be a finite extension of $\mathbb{Q}_p$ with $p > 2$. Let $\bar{\rho} : \Gamma_K \to \mathrm{GL}_n(\mathbb{F})$ be a n-dimensional continuous representation of the absolute Galois group Γ_K over $\mathbb{F}$ for some $n \in \mathbb{N}$, and where $\mathbb{F}$ is a finite field of characteristic p. Kisin constructed in [31] a projective scheme $C_\mu(\bar{\rho})$ over $\mathbb{F}$ which parametrizes the finite flat group schemes over $\mathcal{O}_K$ with generic fiber $\bar{\rho}$ satisfying some determinant condition determined by μ. These varieties were later called *Kisin varieties* by Pappas and Rapoport ([44]). Kisin showed in [31] that the set of connected components $\pi_0(C_\mu(\bar{\rho}))$ of the Kisin variety is in bijection with the set of connected components of the generic fiber of the flat deformation ring of $\bar{\rho}$ with condition on Hodge-Tate weights related to μ. Moreover, Kisin determined $\pi_0(C_\mu(\bar{\rho}))$ when K is totally ramified over $\mathbb{Q}_p$ with $\bar{\rho}$ two dimensional and gave application to modularity lifting theorem for two-dimensional p-adic representations of the Galois group of $\mathbb{Q}$, which is a generalization of the well known Wiles [52] and Taylor-Wiles [49] methods. Extending Kisin's work, Gee also proved a modularity lifting theorem in [17] for totally real field by studying the connected components of some Kisin variety. Recently, Calegari and Geraghty developed new methods and proved modularity lifting theorems for a large number of new cases ([3]).

Besides modularity lifting, there are other motivations for the study of (generalized) Kisin varieties. Emerton and Gee constructed in [13] the moduli stack of (φ, Γ)-modules which is considered to be a breakthrough in the p-adic Langlands program. Some properties of this stack is related to Kisin varieties (see for example [4]). Moreover, in a series of papers ([33], [34], [35], [36]) of Le, Le

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Hung, Levin and Morra on Breuil-Mézard conjecture and weight part of Serre conjecture, they also use the structure of Kisin varieties.

The geometry of Kisin varieties has been studied by many people. For example, the dimension of certain Kisin varieties has been studied by Caruso [6], Hellmann [26] and Imai [28, 29]. In this paper, we are interested in the set of connected components $\pi_0(C_\mu(\bar\rho))$ of the Kisn variety. Kisin proposed in [31, 2.4.16] a conjecture on $\pi_0(C_\mu(\bar\rho))$ when $\mathrm{End}_{\mathbb{F}[\Gamma_K]}(\bar\rho) = \mathbb{F}$ (e.g. $\bar\rho$ is absolutely irreducible). In the case of $n = 2$, the conjecture says precisely that the Kisin variety has at most two connected components and this conjecture has been solved by Gee [17], Hellmann [25], Imai [27] and Kisin [31]. For general n, we know that $C_\mu(\bar\rho)$ consists of at most one point if the ramification index $e(K/\mathbb{Q}_p) < p-1$ by Raynaud's result ([47, 3.3.2]). Besides this result, very little is known about the set of connected components. Erickson and Levin studied in [14] the Harder-Narasimhan theory for Kisin modules and defined a stratification on Kisin varieties by the Harder-Narasimhan polygons. Caruso, David and Mézard studied the the connectedness of the (generalized) Kisin varieties associated to non-trivial Galois types ([7, 8]).

In this paper, we want to study the connectedness of Kisin varieties $C_\mu(\bar\rho)$ for an absolutely irreducible representation $\bar\rho$. In this case, Kisin's conjecture says:

Conjecture 1.1 (Kisin)**.** *If $\bar\rho$ is absolutely irreducible, then $C_\mu(\bar\rho)$ is connected.*

We will prove the conjecture when μ is of a particular form or $n = 3$ with K totally ramified. We also give counterexamples to show Kisin's conjecture does not hold in general.

We first reformulate Kisin variety in a group theoretic way.

Let k be the residue field of K. Let π be a uniformizer of K and let $\pi_n := \pi^{\frac{1}{p^n}}$ be a compatible system of p^n-th root of π for all $n \in \mathbb{N}$. Denote $K_\infty := \cup_n K(\pi_n)$. Then $\bar\rho_{|\Gamma_{K_\infty}}$ is still absolutely irreducible ([1, 3.4.3]). By [53], the absolute Galois group Γ_{K_∞} is canonically isomorphic to the absolute Galois group $\Gamma_{k((u))}$ of the field of Laurent series. Hence the $\mathbb{F}$-representations of Γ_{K_∞} can be described in terms of étale φ-modules over $k \otimes_{\mathbb{F}_p} \mathbb{F}((u))$ (cf.[15]), where the Frobenius φ acts on $k \otimes_{\mathbb{F}_p} \mathbb{F}((u))$ as the identity on $\mathbb{F}$ and as the p-power map on $k((u))$. Let $(N_{\bar\rho}, \Phi_{\bar\rho})$ be the φ-module of rank n over $k \otimes_{\mathbb{F}_p} \mathbb{F}((u))$ associated to the Tate-twist $\bar\rho(-1)_{|\Gamma_{K_\infty}}$. Then

$$(N_{\bar\rho}, \Phi_{\bar\rho}) \simeq ((k \otimes_{\mathbb{F}_p} \mathbb{F}((u)))^n, b\varphi)$$

for some $b \in G(\mathbb{F}((u)))$ with $G = \mathrm{Res}_{k|\mathbb{F}_p}(\mathrm{GL}_n)$. In the following, we will write $C_\mu(b)$ for the Kisin variety $C_\mu(\bar\rho)$.

Let T be a maximal torus of G. Let Y^+ denotes the set of dominant cocharacters of T with respect to a fixed Borel subgroup of G containing T. It's a partially ordered set with the Bruhat order $\le$.

Let $\bar{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$ and $L := \bar{\mathbb{F}}_p((u))$ be the field of Laurent series with coefficient in $\bar{\mathbb{F}}_p$. By Breuil-Kisin classification ([30], [31]), the Kisin variety can be described as follows:

$$C_\mu(b)(\bar{\mathbb{F}}_p) = \{g \in G(L)/G(\mathcal{O}_L) | g^{-1}b\sigma(g) \in \cup_{\substack{\nu\in Y^+ \\ \nu\leq\mu}} G(\mathcal{O}_L)u^\nu G(\mathcal{O}_L)\}$$

where the determinant condition can be interpreted as a dominant cocharacter $\mu \in Y^+$ and $\sigma : G(L) \to G(L)$ is an endomorphism induced from φ. So the Kisin variety $C_\mu(b) \otimes \bar{\mathbb{F}}_p$ can be seen as a closed subvariety in the affine Grassmannian $\mathcal{F}_G = G(L)/G(\mathcal{O}_L)$ which is an ind-scheme over $\bar{\mathbb{F}}_p$. Indeed, the original definition of Kisin variety in [31] is not written in the group theoretic way. It's Pappas and Rapoport who first described Kisin varieties in [44] in the form $C_\mu(b)$ as we introduced. The isomorphism class of $C_\mu(b) \otimes \bar{\mathbb{F}}_p$ depends on the σ-conjugacy class of b in $G(L)$.

The group theoretic definition of Kisin varieties resembles that of affine Deligne-Lusztig varieties which are moduli spaces of p-divisible groups with additional structures. The only difference is the endomorphism σ. For affine Deligne-Lusztig varieties, it's an isomorphism. But it's no longer the case for Kisin varieties. This change makes Kisin varieties much harder to study compared to affine Deligne-Lusztig varieties. Much is known about the structure of affine Deligne-Lusztig varieties by the study of many people, such as the non-emptyness ([17], [20],[23], [32],[38],[46]), the dimension formula ([18], [21], [50], [56]), the set of connected components ([9], [10], [11], [24], [41], [51]) and the set of irreducible components up to group action ([22], [42], [54],[55]). One of the powerful tools to study affine Deligne-Lusztig varieties is the semi-module stratification which arises in a group theoretic way. For example, de Jong and Oort used this stratification to show that each connected component of super-basic affine Deligne-Lusztig variety for GL_n is irreducible ([12]). The second author used this stratification to give a proof of a conjecture of Xinwen Zhu and the first author on the irreducible components of affine Deligne-Lusztig varieties ([42], with another proof using twisted orbital integrals given by Zhou and Zhu [55]).

In this paper, we want to apply this tool to the study of Kisin varieties. We restrict ourselves to the case that $\bar{\rho}$ is absolutely irreducible, that is, b is simple (cf. §3). The reason for making such restriction is that we can use Caruso's classification of simple φ-modules ([5]) to get a good representative in the σ-conjugacy class of b (Proposition 3.3).

Note that $G_{|\bar{\mathbb{F}}_p} \simeq \prod_{\tau\in \mathrm{Hom}(k,\bar{\mathbb{F}}_p)} \mathrm{GL}_n$. An element $\mu \in Y^+$ can be written in the form $\mu = (\mu_\tau)_\tau$ with

$$\mu_\tau = (\mu_\tau(1), \cdots, \mu_\tau(n)) \in \mathbb{Z}^{n,+} := \{(a_1, \cdots, a_n) \in \mathbb{Z}^n | a_1 \geq a_2 \geq \cdots \geq a_n\}.$$

Theorem (4.1, 4.6)**.** *Suppose b is simple and the Kisin variety $C_\mu(b)$ is non-empty, then $C_\mu(b)$ is geometrically connected if one of the following two conditions are satisfied:*

(1) *$\mu = (\mu_\tau)_\tau$ with $\mu_\tau(1) \geq \mu_\tau(2) = \mu_\tau(3) = \cdots = \mu_\tau(n)$ for all $\tau \in \mathrm{Hom}(k, \bar{\mathbb{F}}_p)$;*
(2) *K is totally ramified and $n = 3$ (i.e. $G = \mathrm{GL}_3$).*

The first part of the theorem recovers the main result of [25] when $n = 2$.

We also give counterexamples to show that Kisin's conjecture does not hold in general (cf. §4.3).

As an application of this result, we obtain a connectedness result of the deformation ring of $\bar\rho$ with conditions on Hodge-Tate weights. Suppose $\bar\rho : \Gamma_K \to \mathrm{GL}_n(\mathbb{F})$ is absolutely irreducible and flat. Let R^{fl} be the flat deformation ring of $\bar\rho$ in the sense of Ramakrishna ([45]). Consider a minuscule cocharacter

$$\nu : \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \to (\mathrm{Res}_{K|\mathbb{Q}_p} T_n)_{\bar{\mathbb{Q}}_p},$$

where T_n is the maximal torus of GL_n consisting of diagonal matrices. Write $R^{fl,\nu}$ as the quotient of R^{fl} corresponding to deformations of Hodge-Tate weights given by ν.

Corollary (5.1)**.** *The scheme* $\mathrm{Spec}(R^{\mathrm{fl},\nu}[\frac{1}{p}])$ *is connected if one of the following two conditions holds:*

(1) $\nu_\tau = (1, 0 \cdots, 0)$ *or central for all* $\tau \in \mathrm{Hom}(K, \bar{\mathbb{Q}}_p)$*;*
(2) K *is totally ramified and* $n = 3$*.*

The group theoretic methods in this paper is new in the study of the structure of Kisin varieties. We hope our techniques and results can find applications within the new developments.

We now a give brief outline of the article. In Section 2, we introduce the semi-module stratification which is the main tool for us to study the geometry of Kisin varieties. In Section 3, using Caruso's classification of simple φ-modules, we construct a good representative b in its σ-conjugacy class. In section 4, we prove the main theorem about the connectedness of Kisin varieties and also give some counterexamples to Kisin's conjecture. In section 5, we apply the main theorem to get a connectedness result for deformation rings.

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2. Semi-module stratification on Kisin varieties

In this section, suppose p is a prime number (we allowed $p = 2$).

2.1. **Kisin variety for arbitrary reductive groups.** Recall that $L = \bar{\mathbb{F}}_p((u))$ and $\mathcal{O}_L = \bar{\mathbb{F}}_p[\![u]\!]$. Let $\varphi_L : L \to L$ be the homomorphism given by $\sum_i a_i u^i \mapsto \sum_i a_i u^{pi}$, where $a_i \in \bar{\mathbb{F}}_p$.

Let G be a reductive group over $\mathbb{F}_p$. Fix $T \subseteq B \subseteq G$ with T a maximal torus and B a Borel subgroup of G. Let $\mathcal{R} = (X, \Phi, Y, \Phi^\vee)$ be the associated root datum, where Φ (resp. $\Phi^\vee$) is the set of roots (resp. coroots) of G; X (resp. Y) is the character group (resp. cocharacter group) of T, together with a perfect pairing $\langle, \rangle : X \times Y \to \mathbb{Z}$. Denote by Φ^+ the set of positive roots appearing in B, and by Y^+ the corresponding set of dominant cocharacters. On Y we have the Bruhat order denoted by $\leq$. Namely, for $\chi, \chi' \in Y$ we have $\chi \leq \chi'$ if and only if $\bar{\chi} - \bar{\chi}'$ is sum of positive coroots, where $\bar{\chi}$ and $\bar{\chi}'$ are the unique dominant conjugates of χ and χ' respectively.

Let $G(L)$ be the group of L-points of G. We have the Cartan decomposition on $G(L)$:

$$G(L) = \coprod_{\mu \in Y^+} G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L).$$

It induces a stratification of the affine Grassmannian for G in the strict sense, i.e., for any $\mu \in Y^+$,

$$\overline{G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L)}/G(\mathcal{O}_L) = \coprod_{\substack{\nu \in Y^+ \\ \nu \leq \mu}} G(\mathcal{O}_L) u^\nu G(\mathcal{O}_L)/G(\mathcal{O}_L).$$

Fix an automorphism σ_0 of G that fixes T and B. Denote by σ the endomorphism of $G(L)$ induced from φ and σ_0 as follows:

$$\sigma : G(L) \xrightarrow{\sigma_0} G(L) \xrightarrow{\varphi_L} G(L).$$

Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$. Let $b \in G(\mathbb{F}((u)))$ and $\mu \in Y^+$ such that $\mathbb{F}$ contains the reflex field of the conjugacy class of μ. Then the associated (closed) Kisin variety is defined over $\mathbb{F}$ and with $\bar{\mathbb{F}}_p$-points given by

$$C_\mu(b)(\bar{\mathbb{F}}_p) = \{g \in G(L); g^{-1} b \sigma(g) \in \overline{G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L)}\}/G(\mathcal{O}_L).$$

In the literature, Kisin varieties were defined for the groups of the form $G = \mathrm{Res}_{k|\mathbb{F}_p} H$ and the automorphism σ_0 is induced by the Frobenius relative to $k|\mathbb{F}_p$, where k is finite extension of $\mathbb{F}_p$ and H is a reductive group over k. (cf. [44], [14]). In this case, the Kisin variety has the moduli interpretation that parametrizes some finite flat models with H-structure of a Galois representation with value in H ([37]).

2.2. **Semi-module stratification of Kisin varieties.** In the rest of this section, we are interested in $G \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ and $C_\mu(b) \times_{\mathbb{F}} \bar{\mathbb{F}}_p$ but not their rational structure.

Let $N_T \subseteq G$ be the normalizer of T. Denote by $W_0 = N_T(L)/T(L)$ the Weyl group, and denote by

$$\tilde{W} = Y \rtimes W_0 = \{u^\tau w; \tau \in Y, w \in W_0\}$$

the Iwahori-Weyl group. There is a natural action of $\tilde{W}$ on the vector space $Y_\mathbb{R} = Y \otimes \mathbb{R}$. For $\tilde{w} = u^\tau w \in \tilde{W}$ we set $\dot{\tilde{w}} = u^\tau \dot{w}$ with $\dot{w} \in N_T(\bar{\mathbb{F}}_p)$ some/any lift of w. As σ and σ_0 preserve $T(L)$, we still denote by σ and σ_0 the induced endomorphisms on Y respectively. Notice that σ_0 is an automorphism of the root datum $\mathcal{R}$ and $\sigma = p\sigma_0$.

Lemma 2.1. *Let $\tilde{w} \in \tilde{W}$. The map $\tilde{w}\sigma : Y_\mathbb{R} \to Y_\mathbb{R}$ has a unique fixed point.*

Proof. Suppose $Y_\mathbb{R} = \mathbb{R}^n$ for some $n \geq 0$. Then $\tilde{w}\sigma$ is an affine transformation of the following form:

$$\tilde{w}\sigma(x) = pAx + y, \forall x \in Y_\mathbb{R} = \mathbb{R}^n,$$

where $y \in \mathbb{R}^n$ and $A \in \mathrm{GL}_n(\mathbb{R})$ is of finite order. Hence the eigenvalues of A are all roots of unity and non of the eigenvalues of pA is 1. It follows that $\tilde{w}\sigma$ has a unique fixed point in $Y_{\mathbb{R}}$. □

Let $I \subseteq G(L)$ be the Iwahori subgroup associated to the fundamental alcove

$$\Delta = \{v \in Y_{\mathbb{R}}; 0 < \langle \alpha, v\rangle < 1, \forall \alpha \in \Phi^+\}.$$

Namely, I is the preimage of $B^-(\bar{\mathbb{F}}_p)$ under natural map $G(\mathcal{O}_L) \overset{u \mapsto 0}{\longrightarrow} G(\bar{\mathbb{F}}_p)$, where B^- is the opposite Borel subgroup of B. Denote by

$$I_+ = U(u\mathcal{O}_L)T(1 + u\mathcal{O}_L)U^-(\mathcal{O}_L) \subseteq I$$

be the pro-unipotent radical of I, namely the preimage of $U^-(\bar{\mathbb{F}}_p)$ under the natural map $G(\mathcal{O}_L) \to G(\bar{\mathbb{F}}_p)$. Here U (resp. U^-) is the unipotent radical of B (resp. B^-).

For any $r \in \mathbb{N}$, denote by $T(1 + u^r\mathcal{O}_L)$ the image of $(1 + u^r\mathcal{O}_L)^n \subset \mathbb{G}_m^n(L)$ via an isomorphism of algebraic groups $\mathbb{G}_m^n \simeq T_{|\bar{\mathbb{F}}_p}$, where n is the rank of T. It's easy to see $T(1 + u^r\mathcal{O}_L)$ does not depend on the choice of the isomorphism $\mathbb{G}_m^n \simeq T_{|\bar{\mathbb{F}}_p}$.

Lemma 2.2. *Let $b = \dot{\tilde{w}}$ with $\tilde{w} \in \tilde{W}$ such that the (unique) fixed point of $\tilde{w}\sigma$ lies in Δ. Then the Lang's map*

$$\Psi : G(L) \to G(L)$$
$$h \mapsto h^{-1}b\sigma(h)b^{-1}$$

restricts to a bijection $I_+ \cong I_+$.

Proof. Let $e \in \Delta$ be the fixed point of $\tilde{w}\sigma$ by Lemma 2.1. For $r \in \mathbb{R}_{\geqslant 0}$ we denote by $I_r \subseteq I$ the (normal) Moy-Prasad subgroup (cf. [40]), which is generated by $T(1 + u^{\lceil r\rceil}\mathcal{O}_L)$ and $U_\alpha(u^k\mathcal{O}_L)$ such that $-\langle \alpha, e\rangle + k \geqslant r$. Note that $I = I_0$ and $I_+ = I_r$ for $r > 0$ sufficiently small. Let $\iota : G(L) \to G(L)$ be the map sending h to $b\sigma(h)b^{-1}$. Then $\Psi(h) = h^{-1}\iota(h)$.

Claim: $\iota(I_r) \subseteq I_{pr}$ for $r \geqslant 0$. In particular, the restriction of ι on I_+ is contractive, that is, $\lim_{n\to+\infty} \iota^n(h)$ is the unit element in I_+ for any $h \in I_+$.

The claim implies that Ψ is a bijection. Indeed, if the Claim follows, we can define $\Psi^{-1} : I_+ \to I_+$ such that

$$\Psi^{-1}(h) = (h\iota(h)\iota^2(h)\cdots)^{-1}.$$

This infinite product is well defined because of the Claim. Moreover, it's easy to check that Ψ^{-1} is the inverse of Ψ.

Now we prove the Claim. Write $\tilde{w} = u^\tau w$ with $\tau \in Y$ and $w \in W_0$. Then $e = w\sigma(e) + \tau = pw\sigma_0(e) + \tau$. One computes that

$$b\sigma(T(1 + u^{\lceil r\rceil}\mathcal{O}_L))b^{-1} = T(1 + u^{p\lceil r\rceil}\mathcal{O}_L) \subseteq T(1 + u^{\lceil pr\rceil}\mathcal{O}_L);$$
$$b\sigma(U_\alpha(u^k\mathcal{O}_L))b^{-1} = U_{w\sigma_0(\alpha)}(u^{pk+\langle w\sigma_0(\alpha),\tau\rangle}\mathcal{O}_L) \subseteq I_{pr},$$

where the second inclusion follows from that

$$-\langle w\sigma_0(\alpha), e\rangle + pk + \langle w\sigma_0(\alpha), \tau\rangle = -\langle w\sigma_0(\alpha), pw\sigma_0(e) + \tau\rangle + pk + \langle w\sigma_0(\alpha), \tau\rangle$$
$$= p(-\langle \alpha, e\rangle + k) \geq pr.$$

This finishes the proof. □

For $\lambda \in Y$, let U_λ^+ (resp. U_λ^-) be the subgroup of G generated by U_α such that $\lambda_\alpha \geqslant 0$ (resp. $\lambda_\alpha < 0$), where

$$\lambda_\alpha = \begin{cases} \langle \lambda, \alpha \rangle, & \text{if } \alpha \in \Phi^-; \\ \langle \lambda, \alpha \rangle - 1, & \text{if } \alpha \in \Phi^+. \end{cases}$$

Note that U_λ^+ and U_λ^- are opposite to each other. If there is no confusion, we also write U_λ^+ (resp. U_λ^-) for $U_\lambda^+(L)$ (resp. $U_\lambda^-(L)$).

Lemma 2.3. *The groups $U_\lambda^\pm$ are opposite maximal unipotent subgroups of G. In particular, if $U_\lambda^+ u^\chi$ or $U_\lambda^- u^\chi$ intersects $\overline{G(\mathcal{O}_L) u^{\chi'} G(\mathcal{O}_L)}$ for some $\chi, \chi' \in Y$, then $\chi \leq \chi'$.*

Proof. Let $v \in \Delta$, that is, $0 < \langle \alpha, v \rangle < 1$ for all $\alpha \in \Phi^+$. Let $\beta \in \Phi$. Then we have $\langle \lambda - v, \beta \rangle \neq 0$. Moreover, it follows by definition that $\langle \lambda - v, \beta \rangle > 0$ (resp. $\langle \lambda - v, \beta \rangle < 0$) if and only if $\lambda_\beta \geqslant 0$ (resp. $\lambda_\beta < 0$). This means that $U_\lambda^+ = \prod_{\langle \lambda - v, \beta \rangle > 0} U_\beta$ and $U_\lambda^- = \prod_{\langle \lambda - v, \beta \rangle < 0} U_\beta$ are opposite maximal unipotent subgroups. The second statement follows directly from [43, Lemme 4.2] (where we take $U = U_\lambda^+$, $\nu = \chi$, and $\lambda = \chi'$). □

Consider the following semi-module decomposition

$$C_\mu(b)(\bar{\mathbb{F}}_p) = \sqcup_{\lambda \in Y} C_\mu^\lambda(b)(\bar{\mathbb{F}}_p),$$

where each piece $C_\mu^\lambda(b)$ is locally closed subscheme of $C_\mu(b) \times_{\mathbb{F}} \bar{\mathbb{F}}_p$ with $\bar{\mathbb{F}}_p$-points

$$C_\mu^\lambda(b)(\bar{\mathbb{F}}_p) = (I u^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)) \cap C_\mu(b)(\bar{\mathbb{F}}_p).$$

Proposition 2.4. *Let $b = \dot{\tilde{w}}$, where $\tilde{w} = u^\tau w \in \tilde{W}$ is as in Lemma 2.2 for some $\tau \in Y$ and $w \in W_0$. The following conditions are equivalent:*

(1) *$C_\mu^\lambda(b)$ is non-empty;*
(2) *$u^\lambda \in C_\mu(b)(\bar{\mathbb{F}}_p)$;*
(3) *$\lambda^\natural := -\lambda + \tau + w\sigma(\lambda) \leq \mu$.*

Moreover, under these equivalent conditions we have

(a) $C_\mu^\lambda(b)$ is connected;
(b) $C_\mu(b) = \{u^\lambda\}$ if and only if

$$I \cap U_\lambda^+ \cap u^\lambda \overline{G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger} = u^\lambda U_\lambda^+(\mathcal{O}_L) u^{-\lambda},$$

where $\lambda^\dagger = \tau + w\sigma(\lambda)$;

(c) $C_\mu^\lambda(b)$ is irreducible of dimension $|R(\lambda)|$ if μ is minuscule, where $R(\lambda) = \{\alpha \in \Phi; \lambda_\alpha \geqslant 1, \langle \alpha, \lambda^\natural \rangle = -1\}$.

Proof. We first show the three conditions are equivalent. Note that $u^{-\lambda} b \sigma(u^\lambda) = u^{\lambda^\natural}$. It follows the implications (3) ⇒ (2) ⇒ (1). It remains to show the implication (1) ⇒ (3). Let $\iota : I_+ \to I_+$ be the endomorphism of I_+ given by $h \mapsto b\sigma(h)b^{-1}$. Let $\Psi : I_+ \to I_+$ be the Lang's map given by $h \mapsto h^{-1}\iota(h)$. By Lemma 2.2, Ψ is an isomorphism whose inverse is given by $h \mapsto \cdots \iota^2(h)^{-1}\iota(h)^{-1}h^{-1}$. Denote by $\pi_\lambda : I_+ \to I u^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ the surjective map given by $h \mapsto h u^\lambda G(\mathcal{O}_L)$. Then one computes that

$$\pi_\lambda^{-1}(C_\mu^\lambda(b)(\bar{\mathbb{F}}_p)) = \Psi^{-1}(I_+ \cap u^\lambda \overline{G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger}),$$

with $\lambda^\dagger = \tau + w\sigma(\lambda)$. In view of Lemma 2.3, we have the following Iwahori decomposition (cf. [19, §13.1])

$$I_+ = (I \cap U_\lambda^-)T(1 + u\mathcal{O}_L)(I \cap U_\lambda^+).$$

Noticing that $u^{-\lambda}(I \cap U_\lambda^-)T(1 + u\mathcal{O}_L)u^\lambda \subseteq I$, we deduce that

$$\begin{aligned} &I_+ \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger} \\ = \quad &(I \cap U_\lambda^-)T(1 + u\mathcal{O}_L) \; ((I \cap U_\lambda^+) \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger}). \end{aligned} \tag{2.2.1}$$

Suppose $C_\mu^\lambda(b)(\bar{\mathbb{F}}_p)$ is non-empty, that is,

$$I_+ \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger} \neq \emptyset. \tag{2.2.2}$$

Moreover, notice that $\lambda^\natural = \lambda^\dagger - \lambda$ and

$$u^{-\lambda}(I \cap U_\lambda^+)u^{\lambda^\dagger} = u^{-\lambda}(I \cap U_\lambda^+)u^\lambda u^{\lambda^\natural} \subseteq U_\lambda^+ u^{\lambda^\natural} \tag{2.2.3}$$

as U_λ^+ is normalized by $u^{-\lambda} \in T$. Thus by (2.2.1), (2.2.2) and (2.2.3), we have

$$\emptyset \neq (u^{-\lambda}(I \cap U_\lambda^+)u^{\lambda^\dagger}) \cap \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} \subseteq U_\lambda^+ u^{\lambda^\natural} \cap \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)},$$

which means that $\lambda^\natural \leq \mu$ by Lemma 2.3. So the implication (1) ⇒ (3) is proved.

Now suppose that the three equivalent conditions are satisfied. Choose $n \gg 0$ such that both $I_+ \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger}$ and $I_+ \cap u^\lambda G(\mathcal{O}_L) u^{-\lambda}$ are invariant under the left multiplication by I_n, see the proof of Lemma 2.2. Then Ψ induces an automorphism

$$\Psi_n : I_+/I_n \xrightarrow{\sim} I_+/I_n.$$

Now we have

$$\begin{aligned} C_\mu^\lambda(b)(\bar{\mathbb{F}}_p) &= \pi_\lambda \circ \Psi^{-1}(I_+ \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger}) \\ &\cong \Psi^{-1}(I_+ \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger})/I_+ \cap u^\lambda G(\mathcal{O}_L) u^{-\lambda} \\ &\cong \Psi_n^{-1}(I_+ \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger}/I_n)/((I_+ \cap u^\lambda G(\mathcal{O}_L) u^{-\lambda})/I_n). \end{aligned}$$

As Ψ_n is an isomorphism, to show $C_\mu^\lambda(b)$ is connected, it suffices to show

$$I_+ \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger}/I_n$$

is connected. In view of (2.2.1) and that $(I \cap U_\lambda^-)T(1 + u\mathcal{O}_L)$ is connected, this is equivalent to show

$$(I \cap U_\lambda^+) \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger}/I_n \cap U_\lambda^+$$

is connected. Let ξ be a regular dominant cocharacter with respect to U_λ^+. Then the map $z \mapsto z^\xi h z^{-\xi}$ defines an affine line connecting an arbitrary point $h \in (I \cap U_\lambda^+) \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger}/I_n \cap U_\lambda^+$ and the identity 1. This finishes the proof of (a).

One computes that

$$\begin{aligned}
(2.2.4)\quad & \dim C^\lambda_\mu(b) \\
&= \dim(I_+ \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger})/I_n - \dim(I_+ \cap u^\lambda G(\mathcal{O}_L)u^{-\lambda})/I_n \\
&= \dim(I \cap U^-_\lambda)T(1+u\mathcal{O}_L)((I\cap U^+_\lambda) \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger})/I_n \\
&\quad - \dim(I \cap U^-_\lambda)T(1+u\mathcal{O}_L)(u^\lambda U^+_\lambda(\mathcal{O}_L)u^{-\lambda})/I_n \\
&= \dim(I\cap U^+_\lambda) \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger})/I_n \cap U^+_\lambda \\
&\quad - \dim(u^\lambda U^+_\lambda(\mathcal{O}_L)u^{-\lambda})/I_n \cap U^+_\lambda,
\end{aligned}$$

where the second equality follows from (2.2.1) and

$$I_+ \cap u^\lambda G(\mathcal{O}_L)u^{-\lambda} = (I\cap U^-_\lambda)T(1+u\mathcal{O}_L)(u^\lambda U^+_\lambda(\mathcal{O}_L)u^{-\lambda}).$$

By (a), $C^\lambda_\mu(b) = \{u^\lambda\}$ if and only if $\dim C^\lambda_\mu(b) = 0$, that is,

$$I\cap U^+_\lambda \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger} = u^\lambda U^+_\lambda(\mathcal{O}_L)u^{-\lambda}$$

since

$$u^\lambda U^+_\lambda(\mathcal{O}_L)u^{-\lambda} \subseteq I\cap U^+_\lambda \cap u^\lambda \overline{G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger}$$

with both sides connected and are invariant under left multiplication by $u^\lambda U^+_\lambda(\mathcal{O}_L)u^{-\lambda}$. Thus (b) is proved.

Suppose μ is minuscule. Then we have

Claim: $U^+_\lambda u^{\lambda^\natural} \cap G(\mathcal{O}_L)u^{\lambda^\natural}G(\mathcal{O}_L) = U^+_\lambda(\mathcal{O}_L)u^{\lambda^\natural}U^+_\lambda(\mathcal{O}_L)$.

Indeed, as μ is minuscule, it follows from (3) that $\lambda^\natural = w(\mu)$ for some $w \in W_0$. By [43, Lemme 5.2] (where we take $U = U^+_\lambda$ and $\lambda = \mu$) or [39, (3.6)],

$$U^+_\lambda u^{\lambda^\natural} G(\mathcal{O}_L) \cap G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L) = U^+_\lambda(\mathcal{O}_L)u^{\lambda^\natural}G(\mathcal{O}_L).$$

So the Claim follows by noticing that $G(\mathcal{O}_L) \cap U^+_\lambda = U^+_\lambda(\mathcal{O}_L)$.

Now we have

$$U^+_\lambda u^{\lambda^\natural} \cap G(\mathcal{O}_L)u^\mu G(\mathcal{O}_L) = U^+_\lambda(\mathcal{O}_L)u^{\lambda^\natural}U^+_\lambda(\mathcal{O}_L) = U^+_\lambda(\mathcal{O}_L)(\prod_{\alpha\in D} U_\alpha(t^{-1}\mathcal{O}_L))u^{\lambda^\natural},$$

where $D = \{\alpha \in \Phi; \lambda_\alpha \geqslant 0, \langle \alpha, \lambda^\natural\rangle = -1\}$. As $\lambda^\natural$ is minuscule, all the root subgroups U_α for $\alpha \in D$ commute with each other. Indeed, if U_α and $U_{\alpha'}$ doesn't commute for $\alpha, \alpha' \in D$, then by [48, 8.2.3], there exists $i, i' \in \mathbb{N}_{>0}$ such that $i\alpha + i'\alpha' \in \Phi$. It follows that $\langle i\alpha + i'\alpha', \lambda^\natural\rangle = -i - i' \leq -2$ which contracts the fact that $\lambda^\natural$ is minuscule. It follows that

$$
\begin{aligned}
(2.2.5) \qquad & (I \cap U_\lambda^+) \cap u^\lambda \overline{G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger} \\
&= (I \cap U_\lambda^+) \cap u^\lambda G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L) u^{-\lambda^\dagger} \\
&= u^\lambda (u^{-\lambda} (I \cap U_\lambda^+) u^\lambda u^{\lambda^\natural} \cap G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L)) u^{-\lambda^\dagger} \\
&= u^\lambda ((u^{-\lambda} (I \cap U_\lambda^+) u^\lambda u^{\lambda^\natural}) \cap (U_\lambda^+(\mathcal{O}_L)(\prod_{\alpha \in D} U_\alpha(u^{-1}\mathcal{O}_L)) u^{\lambda^\natural})) u^{-\lambda^\dagger} \\
&= I \cap ((u^\lambda U_\lambda^+(\mathcal{O}_L) u^{-\lambda}) \prod_{\alpha \in D} U_\alpha(u^{\langle \alpha, \lambda \rangle - 1} \mathcal{O}_L) \\
&= (u^\lambda U_\lambda^+(\mathcal{O}_L) u^{-\lambda})(I \cap (\prod_{\alpha \in D} U_\alpha(u^{\langle \alpha, \lambda \rangle - 1} \mathcal{O}_L)) \\
&= (u^\lambda U_\lambda^+(\mathcal{O}_L) u^{-\lambda}) \prod_{\alpha \in D} I \cap U_\alpha(u^{\langle \alpha, \lambda \rangle - 1} \mathcal{O}_L) \\
&= (u^\lambda U_\lambda^+(\mathcal{O}_L) u^{-\lambda}) \prod_{\alpha \in D \setminus R(\lambda)} U_\alpha(u^{\langle \alpha, \lambda \rangle} \mathcal{O}_L) \prod_{\alpha \in R(\lambda)} U_\alpha(u^{\langle \alpha, \lambda \rangle - 1} \mathcal{O}_L) \\
&= (u^\lambda U_\lambda^+(\mathcal{O}_L) u^{-\lambda}) \prod_{\alpha \in R(\lambda)} U_\alpha(u^{\langle \alpha, \lambda \rangle - 1} \overline{\mathbb{F}}_p),
\end{aligned}
$$

where the fifth equality follows from that $u^\lambda U_\lambda^+(\mathcal{O}_L) u^{-\lambda} \subseteq I$; the sixth one follows from that $I \cap U_\lambda^+ = \prod_{\alpha \in \Phi, \lambda_\alpha \geqslant 0} (I \cap U_\alpha)$; the last but one equality follows from that

$$
I \cap U_\alpha(u^{\langle \lambda, \alpha \rangle - 1} \mathcal{O}_L) = \begin{cases} U_\alpha(u^{\langle \lambda, \alpha \rangle - 1} \mathcal{O}_L), & \text{if } \lambda_\alpha \geqslant 1; \\ U_\alpha(u^{\langle \lambda, \alpha \rangle} \mathcal{O}_L), & \text{if } \lambda_\alpha = 0. \end{cases}
$$

In view of (2.2.1) and (2.2.5), $I_+ \cap u^\lambda \overline{G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger} / I_n$ is irreducible. Hence $C_\mu^\lambda(b)$ is irreducible. For the dimension of $C_\mu^\lambda(b)$, we continue the computation that we started in (2.2.4).

$$
\begin{aligned}
\dim C_\mu^\lambda(b) &\overset{(2.2.4)}{=} \dim (I \cap U_\lambda^+) \cap u^\lambda \overline{G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger}) / I_n \cap U_\lambda^+ \\
&\quad - \dim (u^\lambda U_\lambda^+(\mathcal{O}_L) u^{-\lambda}) / I_n \cap U_\lambda^+ \\
&\overset{(2.2.5)}{=} \dim \prod_{\alpha \in R(\lambda)} U_\alpha(u^{\langle \alpha, \lambda \rangle - 1} \overline{\mathbb{F}}_p) \\
&= |R(\lambda)|.
\end{aligned}
$$

This finishes the proof of (c). □

2.3. **Multi-copy case.** Let G^d be the product of d copies of G. For $b \in G(L)$ and $\mu_\bullet \in Y^d$ we define $C_{\mu_\bullet}(b)$ over $\bar{\mathbb{F}}_p$ with $\bar{\mathbb{F}}_p$-points:

$$
C_{\mu_\bullet}(b)(\bar{\mathbb{F}}_p) = \{g_\bullet \in G^d(L); g_\bullet^{-1} b_\bullet \sigma_\bullet(g_\bullet) \in \overline{G^d(\mathcal{O}_L) u^{\mu_\bullet} G^d(\mathcal{O}_L)}\} / G^d(\mathcal{O}_L),
$$

where $b_\bullet = (1, \dots, 1, b) \in G^d(L)$ and $\sigma_\bullet : G^d(L) \to G^d(L)$ is the endomorphism given by $(g_1, g_2, \dots, g_d) \mapsto (g_2, \dots, g_d, \sigma(g_1))$. We still denote by $\sigma_\bullet$ the induced linear map $Y_\mathbb{R}^d \to Y_\mathbb{R}^d$ given by $(v_1, v_2, \dots, v_d) \mapsto (v_2, \dots, v_d, \sigma(v_1))$.

Lemma 2.5. *Let $\mu_\bullet = (\mu_1, \dots, \mu_d)$ be a dominant cocharacter and let $\mu = \mu_1 + \cdots + \mu_d$. Then the projection $G^d(L) \to G(L)$ to the first factor induces a surjection $C_{\mu_\bullet}(b)(\bar{\mathbb{F}}_p) \twoheadrightarrow C_\mu(b)(\bar{\mathbb{F}}_p)$.*

Proof. Note that

$$\overline{G(\mathcal{O}_L)u^{\mu_1}G(\mathcal{O}_L)} \cdot \overline{G(\mathcal{O}_L)u^{\mu_2}G(\mathcal{O}_L)} \cdots \overline{G(\mathcal{O}_L)u^{\mu_d}G(\mathcal{O}_L)} = \overline{G(\mathcal{O}_L)u^{\mu}G(\mathcal{O}_L)}$$

by [2] 4.4.4. The lemma follows by direct computation. □

Suppose $b = \dot{\tilde{w}}$ for some $\tilde{w} = u^\tau w \in \tilde{W}$. Set $\tilde{w}_\bullet = (1, \dots, 1, \tilde{w}) = u^{\tau_\bullet} w_\bullet$ with $\tau_\bullet = (0, \dots, 0, \tau)$ and $w_\bullet = (1, \dots, 1, w)$. Let $\lambda_\bullet \in Y^d$. Define

$$\lambda_\bullet^\natural = -\lambda_\bullet + \tau_\bullet + w_\bullet \sigma_\bullet(\lambda_\bullet);$$

$$R(\lambda_\bullet) = \coprod_{i=1}^{d} \{\alpha \in \Phi; (\lambda_i)_\alpha \geqslant 1, \langle \alpha, (\lambda_\bullet^\natural)_i \rangle = -1\}.$$

Moreover, we set $C_{\mu_\bullet}^{\lambda_\bullet}(b)(\bar{\mathbb{F}}_p) = I^d u^{\lambda_\bullet} G^d(\mathcal{O}_L)/G^d(\mathcal{O}_L) \cap C_{\mu_\bullet}(b)(\bar{\mathbb{F}}_p)$.

Proposition 2.6. *Let $\mu_\bullet \in Y^d$ be minuscule, and let $b = \dot{\tilde{w}}$ be as in Lemma 2.2. Then $C_{\mu_\bullet}^{\lambda_\bullet}(b) \neq \emptyset$ if and only if $\lambda_\bullet^\natural$ is conjugate to $\mu_\bullet$. Moreover, in this case, $C_{\mu_\bullet}^{\lambda_\bullet}(b)$ is irreducible of dimension $|R(\lambda_\bullet)|$.*

Proof. By Lemma 2.2, the endomorphism of I_+^d given by $g_\bullet \mapsto b_\bullet \sigma_\bullet(g_\bullet) b_\bullet^{-1}$ is contractible. Hence the Lang's map $I_+^d \to I_+^d$ given by $g_\bullet \mapsto g_\bullet^{-1} b_\bullet \sigma_\bullet(g_\bullet) b_\bullet^{-1}$ is a bijection. Moreover, the fixed point of $\tilde{w}_\bullet \sigma_\bullet$ lies in the fundamental alcove of $Y_{\mathbb{R}}^d$. Then the statement follows the same way as Proposition 2.4 (c). □

3. Classification of simple φ-modules

Suppose $G = \mathrm{Res}_{k|\mathbb{F}_p} H$ with $k = \mathbb{F}_q$ and H is a reductive group over k, equipped with σ_0 induced from the Frobenius relative to $k|\mathbb{F}_p$. We say $b \in G(L)$ is *simple* if $b \notin P(L)$ for any σ-stable proper parabolic subgroup P of G up to σ-conjugation. It's equivalent to say the φ-module

$$((k \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p((u)))^n, b\varphi)$$

over $k \otimes \bar{\mathbb{F}}_p((u))$ is irreducible, where φ is as in the introduction. Let $B(G)$ be the set of σ-conjugacy classes in $G(L)$. A σ-conjugacy class $[b] \in B(G)$ is called simple if all/some representative in this σ-conjugacy class is simple. Let $B(G)_s$ be the subset of $B(G)$ consisting of simple elements. Notice that $C_\mu(b) \times_{\mathbb{F}} \bar{\mathbb{F}}_p$ depends on the $[b] \in B(G)$. The goal of this section is to give a good representative of b in its σ-conjugacy class.

There is a natural identification $G(L) = \prod_{i=1}^{f} H(L)$ with $f = [k : \mathbb{F}_p]$. Under this identification, we have

$$\begin{aligned} \sigma : G(L) &\longrightarrow G(L) \\ (x_1, \cdots, x_f) &\longmapsto (\varphi_L(x_2), \cdots, \varphi_L(x_f), \varphi_L(x_1)). \end{aligned}$$

For the group H over k, we define $T_H \subset B_H$ maximal torus and Borel subgroup of H such that $T = \mathrm{Res}_{k|\mathbb{F}_p} T_H$ and $B = \mathrm{Res}_{k|\mathbb{F}_p} B_H$. We also define

$W_{0,H}$, X_H, Y_H, Δ_H, Φ_H, Φ_H^+ in the same way for H. Then there is natural identifications

$$Y \simeq \prod_{i=1}^{f} Y_H \text{ and } X \simeq \prod_{i=1}^{f} X_H.$$

The endomorphism $\sigma : Y \to Y$ is given by $(h_1, h_2, \dots, h_f) \mapsto (ph_2, \dots, ph_f, ph_1)$.

Using the Frobenius relative to $\bar{\mathbb{F}}_p|k$ instead of the Frobenius relative to $\bar{\mathbb{F}}_p|\mathbb{F}_p$, we define similarly $\sigma_H : H(L) \to H(L)$ and simple elements in $H(L)$. We also define $B(H)$ and $B(H)_s$ in the same way.

We can easily check the following lemma.

Lemma 3.1. *There is a Shapiro bijection*

$$B(H) \simeq B(G)$$
$$[b'] \mapsto [b],$$

where $b = (b', 1, \cdots, 1) \in G(L) = \prod_{i=1}^{f} H(L)$. Moreover, it induces a bijection on the subset of simple elements $B(H)_s \simeq B(G)_s$.

From now on, suppose $H = \mathrm{GL}_n$. Let T_H (resp. B_H) be the subgroup of diagonal (resp. upper triangular) matrices. We have the following identifications:

- $X_H = \oplus_{i=1}^n \mathbb{Z}e_i \simeq \mathbb{Z}^n$, $Y_H = \oplus_{i=1}^n \mathbb{Z}e_i^\vee \simeq \mathbb{Z}^n$ with the pairing $X_H \times Y_H \to \mathbb{Z}$ induced by $\langle e_i, e_j^\vee \rangle = \delta_{i,j}$;
- $\Phi = \sqcup_{i=1}^f \Phi_H$, where $\Phi_n = \{\alpha_{i,j} = e_i - e_j; 1 \leqslant i \neq j \leqslant n\}$;
- $\Phi^+ = \sqcup_{i=1}^f \Phi_H^+$, where $\Phi_H^+ = \{\alpha_{i,j}; 1 \leqslant i < j \leqslant n\}$;
- $W_0 = W_{0,H}^f$ with $W_{0,H} = \mathfrak{S}_n$;
- $\Delta = \Delta_H^f$, with $\Delta_H = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_1 > x_2 > \cdots > x_n > x_1 - 1\}$.

We will need the following result of Caruso.

Proposition 3.2 (Caruso)**.** *Suppose $[b] \in B(H)$ is simple. Then there exists a representative $b = au^{\tau'}w' \in [b]$ where*

1. *$a \in H(\bar{\mathbb{F}}_p)$ is central,*
2. *$w' \in W_{0,H} = \mathfrak{S}_n$ is the cyclic permutation given by $i \mapsto i + 1 \mod n$ for $1 \leq i \leq n$;*
3. *$\tau' = (m, 0, \cdots, 0) \in Y_H = \mathbb{Z}^n$ such that $\frac{m(p^{n'}-1)}{p^n-1} \notin \mathbb{Z}$ for any $n'|n$ with $1 \leq n' < n$.*

Proof. By [5, Corollaire 8], we may assume $b = a'u^{\tau'}w'$ where w' as in (2), $a' = \mathrm{diag}(a'_1, 1, \cdots, 1)$ with $a'_1 \in \bar{\mathbb{F}}_p^*$ and $\tau' = (m, 0, \cdots, 0)$. Let $a = \mathrm{diag}(c, \cdots, c)$ with $c \in \bar{\mathbb{F}}_p^*$ such that $c^n = a'_1$. It's easy to check $g^{-1}b\sigma_H(g) = au^{\tau'}w'$, where $g = \mathrm{diag}(c^{n-1}, \cdots, c, 1)$. So (1) is also satisfied. The condition (3) follows from [5, Proposition 3] as a is central. □

Proposition 3.3. *If $b \in G(L)$ is simple, then it is σ-conjugate to some lift $\dot{\tilde{w}}$ of $\tilde{w} \in \tilde{W}$ such that the fixed point $\tilde{w}\sigma$ lies in $\Delta \cap (\mathbb{R} \setminus \mathbb{Z})^{fn} \subseteq \mathbb{R}^{fn} = Y_{\mathbb{R}}$.*

Proof. By Lemma 3.1, any simple element in B(G) is of the form $[b]$, where $b = (b', 1, \cdots, 1)$ and b' is a lift of $\tilde{w}' = u^{\tau'}w'$, where $\tau' = (m, 0, \dots, 0) \in Y_H = \mathbb{Z}^n$ and $w' \in W_{0,H} = \mathfrak{S}_n$ as in Proposition 3.2. Then b is a lift of $\tilde{w}_1 := u^\tau w$ where

$\tau = (\tau', 0, \cdots, 0) \in Y$ and $w = (w', 1, \cdots, 1) \in W$. Let e (resp. e') be the fixed point of $\tilde{w}_1\sigma$ on $Y_{\mathbb{R}} = \mathbb{R}^{fn}$ (resp. $\tilde{w}'\sigma_H$ on $Y_{H,\mathbb{R}} = \mathbb{R}^n$) by Lemma 2.1.

Claim: $e \in (\mathbb{R}\backslash\mathbb{Z})^{fn}$. Moreover, e lies in some alcove Δ_1 in $Y_{\mathbb{R}}$.

By the description of b', we can compute that

$$e' = -(\frac{m}{p^n-1}, \frac{mp}{p^n-1}, \dots, \frac{mp^{n-1}}{p^n-1}).$$

Moreover

$$\frac{mp^i}{p^n-1} - \frac{mp^j}{p^n-1} \notin \mathbb{Z} \text{ for } 0 \leqslant i < j \leqslant n \tag{3.0.1}$$

by the condition (3) in Proposition 3.2 combined with the easy fact that

$$\gcd(p^{a_1}-1, p^{a_2}-1) = p^{\gcd(a_1,a_2)} - 1$$

for any positive integers a_1 and a_2. In particular, taking $i = j-1$ in (3.0.1) implies $\frac{mp^{j-1}}{p^n-1} \notin \mathbb{Z}$ (for $1 \leqslant j \leqslant n$) and hence $e' \in (\mathbb{R} \setminus \mathbb{Z})^n$. Note that the action of $(b\sigma)^f$ on $G(L) = \prod_{i=1}^f H(L)$ stablizes each component of $H(L)$ and its restriction to the first component equals to $b'\sigma_H$. It follows that

$$e = (e', pe', \cdots, p^{f-1}e').$$

This implies the Claim.

Let $z \in \tilde{W}$ such that $\Delta_1 = z(\Delta)$ and let $\tilde{w} = z^{-1}\tilde{w}_1\sigma(z)$. Then the fixed point $z^{-1}(e')$ of $\tilde{w}\sigma$ lies in Δ as desired. □

4. Main results

In this section, let $G = \mathrm{Res}_{k|\mathbb{F}_p}\mathrm{GL}_n$. Then $G_{\bar{\mathbb{F}}_p} \simeq (\mathrm{GL}_n)^f$ with $f = [k : \mathbb{F}_p]$. We will prove two connectedness results for the Kisin variety $C_\mu(b)$.

4.1. The first result is the following for μ of particular form.

Theorem 4.1. *Let $b \in G(L)$ be simple and $\mu = (\mu_1, \cdots, \mu_f) \in (\mathbb{Z}^n)^f$ such that $\mu_i(1) \geq \mu_i(2) = \cdots = \mu_i(n)$ for all $1 \leq i \leq f$. Then $C_\mu(b)$ is geometrically connected if it is nonempty.*

Remark 4.2. *Theorem 4.1 is proved in* [25, Theorem 1,1] *for $n = 2$.*

In order to prove this theorem, we need a connectedness result for Kisin varieties in the multi-copy case (cf.§2.3). For any positive integer d, consider the group G^d. Let $N = df$. We fix two identifications:

$$\begin{aligned} Y^d &\simeq (\mathbb{Z}^n)^N \\ v_\bullet = (v_1, \dots, v_d) &\mapsto (v^1, \cdots, v^N), \end{aligned} \tag{4.1.1}$$

where $v^{i+(j-1)d} = v_{i,j}$ for any $1 \leq i \leq d$ and $1 \leq j \leq f$ with $v_i = (v_{i,1}, \dots, v_{i,f}) \in Y = (\mathbb{Z}^n)^f$ and

$$W_0^d \simeq (\mathfrak{S}_n)^N$$

$$w_\bullet = (w_1, \dots, w_d) \mapsto (w^1, \cdots, w^N)$$

where $w^{i+(j-1)d} = w_{i,j}$ with $w_i = (w_{i,1}, \dots, w_{i,f}) \in W_0 = \mathfrak{S}_n^f$.

For $v_\bullet \in Y^d$ and $w_\bullet \in W_0^d$, we also define $v^{N+1} := v^1$ and $w^{N+1} := w^1$.

Let $\omega_1^\vee = (1, 0, \dots, 0) \in \mathbb{Z}^n$. Let $\mu_\bullet \in Y^d$. Write $\mu_\bullet = (\mu^1, \dots, \mu^N) \in (\mathbb{Z}^n)^N$ via the above fixed identification $Y^d = (\mathbb{Z}^n)^N$ (4.1.1), and suppose that $\mu^k = m^k \omega_1^\vee$ with $m^k \in \{0, 1\}$ for all $1 \leqslant k \leqslant N$. Let $b \in G(L)$ be as in Lemma 2.2. We consider the variety $C_{\mu_\bullet}(b)$ introduced in §2.3.

Theorem 4.3. *Let $\mu_\bullet$ and b be as above such that $C_{\mu_\bullet}(b) \neq \emptyset$. Then there exists a unique $\lambda_\bullet \in Y^d$ such that $\dim C_{\mu_\bullet}^{\lambda_\bullet}(b) = 0$. Moreover, $C_{\mu_\bullet}(b)$ is connected.*

We first use this result to prove Theorem 4.1

Proof of Theorem 4.1. First note that for any $\mu' \in Y^+$ and any central cocharacter $\chi' \in Y$ we have the identification

$$C_{\mu'}(b) = C_{\mu'+\chi'}(u^{\chi'} b).$$

Set $\omega_n^\vee = (1, \dots, 1) \in \mathbb{Z}^n$. Let $\chi = (-\mu_1(n)\omega_n^\vee, \dots, -\mu_f(n)\omega_n^\vee) \in Y$ be a central cocharacter. Then by replacing μ and b with $\mu+\chi$ and $u^\chi b$ respectively (and using our assumption on μ), we may assume that $\mu = (m_1\omega_1^\vee, \dots, m_f\omega_1^\vee)$ with $m_j \in \mathbb{Z}_{\geqslant 0}$ for $1 \leqslant j \leqslant f$. Thus there exist $d \in \mathbb{Z}_{\geqslant 1}$ and $\mu_i = (\mu_{i,1}, \dots, \mu_{i,f}) \in Y = (\mathbb{Z}^n)^f$ for $1 \leqslant i \leqslant d$ such that $\mu_{i,j} \in \{0, \omega_1^\vee\} \subseteq \mathbb{Z}^n$ for all $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant f$. Let $\mu_\bullet = (\mu_1, \dots, \mu_d) \in Y^d$, which satisfies the assumption of Theorem 4.3.

By Proposition 3.3, we may assume further that b satisfies the condition in Lemma 2.2. Now it follows from Theorem 4.3 that $C_{\mu_\bullet}(b)$ is connected. So $C_\mu(b)$ is also connected as there is a surjective map $C_{\mu_\bullet}(b) \to C_\mu(b)$ by Lemma 2.5. □

Now it remains to prove Theorem 4.3. We need some combinatorial preparations.

For any $v = (v(1), \cdots, v(n)) \in \mathbb{R}^n$, let

- $[v] = \{v(k); 1 \leqslant k \leqslant n\} \subseteq \mathbb{R}$,
- $\langle v \rangle = v(1) + \cdots + v(n) \in \mathbb{R}$,
- $\delta(v) = \langle v \rangle - n \min[v] \in \mathbb{R}$,
- $h(v) = \sum_{i=1}^n \lfloor v(i) - \min[v] \rfloor \in \mathbb{Z}_{\geqslant 0}$.

Note that $h(v) \leq \delta(v) < h(v) + n - 1$. Moreover, $h(v) = 0$ if and only if $\max[v] - \min[v] < 1$.

Define $\varsigma(v) \in \mathbb{R}^n$ such that

$$\varsigma(v)(i) = \begin{cases} v(i) - 1, & \text{if } v(i) = \max[v] \\ v(i), & \text{otherwise.} \end{cases}$$

For $l \in \mathbb{Z}_{>0}$, set $\varsigma^l(v) = \varsigma \circ \cdots \circ \varsigma(v)$, where ς appears l times.

Lemma 4.4. *Let $e \in \mathbb{R}^n$ such that $e(i) - e(j) \notin \mathbb{Z}$ for $1 \leqslant i < j \leqslant n$. Let $v, v' \in \mathbb{Z}^n - e$ such that $\langle v \rangle = \langle v' \rangle$.*

(1) $\delta(\varsigma(v)) = \delta(v) - 1$ and $h(\varsigma(v)) = h(v) - 1$ if $h(v) \geqslant 1$;
(2) $h(\varsigma(v)) = 0$ if $h(v) = 0$;
(3) $v = v'$ if $h(v) = h(v') = 0$;
(4) $\delta(v) \leqslant \delta(v')$ if and only if $h(v) \leqslant h(v')$;
(5) $\delta(\varsigma(v)) \leqslant \delta(\varsigma(v'))$ if $\delta(v) \leqslant \delta(v')$;

Proof. Notice that $[v]$ consists of exactly n elements. Then statements (1) and (2) follow by definition.

(3). Suppose $h(v) = h(v') = 0$, that is, $\max[v]-\min[v], \max[v']-\min[v'] < 1$. Note that $v - v' \in \mathbb{Z}^n$ and $\langle v - v' \rangle = 0$. If $v - v' \neq 0$, then there exist $1 \leqslant i \neq j \leqslant n$ such that $v(i) - v'(i), v'(j) - v(j) \in \mathbb{Z}_{\geqslant 1}$. Thus

$$v(i) - v(j) = (v(i) - v'(i)) + (v'(i) - v'(j)) + (v'(j) - v(j)) > 1 - 1 + 1 = 1,$$

which is a contradiction.

(4). Suppose $h(v) \leqslant h(v')$. By (1) we may replace v, v' with $\varsigma^{h(v)}(v), \varsigma^{h(v)}(v')$ respectively so that $h(v) = 0$. In particular, $\delta(v) < n - 1$. To show $\delta(v) \leqslant \delta(v')$ we may assume $h(v') \leqslant \delta(v') < n - 1$. By (1) and (2) we have

$$h(\varsigma^{h(v')}(v)) = h(\varsigma^{h(v')}(v')) = 0,$$

which means $\varsigma^{h(v')}(v) = \varsigma^{h(v')}(v')$ by (3). Write

$$[\varsigma^{h(v')}(v)] = [\varsigma^{h(v')}(v')] = \{x_1, \dots, x_n\},$$

where $x_1 < x_2 < \cdots < x_n \in \mathbb{R}$. As $h(v) = 0$, we deduce that

$$[v] = \{x_{h(v')+1}, x_{h(v')+2}, \dots, x_n, x_1 + 1, x_2 + 1, \dots, x_{h(v')} + 1\},$$

which means that

$$\delta(v) = \langle \varsigma^{h(v')}(v') \rangle - n x_{h(v')+1} + h(v') \leqslant \langle \varsigma^{h(v')}(v') \rangle - n x_1 + h(v') = \delta(v'),$$

where the equality holds if and only if $h(v') = 0 = h(v)$. This finishes the proof of (4).

(5). If $\delta(v) \leqslant \delta(v')$, then $h(v) \leqslant h(v')$ by (4). So $h(\varsigma(v)) \leqslant h(\varsigma(v'))$ by (1) and (2), which means $\delta(\varsigma(v)) \leqslant \delta(\varsigma(v'))$ by (4). □

Let $\sigma_\bullet : G^d(L) \to G^d(L)$ be the endomorphism defined in §2.3. Then the induced linear map $\sigma_\bullet : (\mathbb{R}^n)^N = Y^d_{\mathbb{R}} \to Y^d_{\mathbb{R}} = (\mathbb{R}^n)^N$ is given by

$$(v^1, v^2, \dots, v^N) \mapsto (\epsilon^1 v^2, \dots \epsilon^{N-1} v^N, \epsilon^N v^1),$$

where $\epsilon^k = p$ if $k \in d\mathbb{Z}$ and $\epsilon^k = 1$ otherwise.

Lemma 4.5. *Let $\mu_\bullet$ and b be as in Theorem 4.3. Let $\lambda_\bullet, \eta_\bullet \in Y^d$ such that $C^{\lambda_\bullet}_{\mu_\bullet}(b)$ and $C^{\eta_\bullet}_{\mu_\bullet}(b)$ are both non-empty. Then $\langle \lambda^k \rangle = \langle \eta^k \rangle$ for any $1 \le k \le N$.*

Proof. By Proposition 2.6, $-\lambda_\bullet + \tau_\bullet + w_\bullet \sigma_\bullet(\lambda_\bullet)$ and $-\eta_\bullet + \tau_\bullet + w_\bullet \sigma_\bullet(\eta_\bullet)$ are both conjugate to $\mu_\bullet$. In particular, for any $1 \le k \le N$,

$$\langle \lambda^k \rangle - \langle \eta^k \rangle = \epsilon^k (\langle \lambda^{k+1} \rangle - \langle \eta^{k+1} \rangle).$$

Therefore

$$\langle \lambda^k \rangle - \langle \eta^k \rangle = (\prod_{i=1}^N \epsilon^i)(\langle \lambda^k \rangle - \langle \eta^k \rangle) = p^f (\langle \lambda^k \rangle - \langle \eta^k \rangle),$$

and the result follows. □

Proof of Theorem 4.3. We first show the existence of $\lambda_\bullet$ with $\dim C^{\lambda_\bullet}_{\mu_\bullet}(b) = 0$. Indeed, let $\lambda_\bullet \in Y^d$ be such that $C^{\lambda_\bullet}_{\mu_\bullet}(b) \neq \emptyset$ and $d_{\lambda_\bullet} := \dim I^d u^{\lambda_\bullet} G^d(\mathcal{O}_L)/G^d(\mathcal{O}_L)$ is as small as possible. If $\dim C^{\lambda_\bullet}_{\mu_\bullet}(b) > 0$, then $C^{\lambda_\bullet}_{\mu_\bullet}(b)$ is irreducible of dimension $\geqslant 1$ by Proposition 2.6. So $C^{\lambda_\bullet}_{\mu_\bullet}(b)$ is a closed (non-projective) subvariety of the affine space $I^d u^{\lambda_\bullet} G^d(\mathcal{O}_L)/G^d(\mathcal{O}_L)$, whose closure $\overline{C^{\lambda_\bullet}_{\mu_\bullet}(b)}$ must intersect $I^d u^{\chi_\bullet} G^d(\mathcal{O}_L)/G^d(\mathcal{O}_L)$ for some $\chi_\bullet \neq \lambda_\bullet$. So $C^{\chi_\bullet}_{\mu_\bullet}(b_\bullet) \neq \emptyset$ and $d_{\chi_\bullet} < d_{\lambda_\bullet}$ since

$$I^d u^{\chi_\bullet} G^d(\mathcal{O}_L)/G^d(\mathcal{O}_L) \subseteq \overline{I^d u^{\lambda_\bullet} G^d(\mathcal{O}_L)/G^d(\mathcal{O}_L)}.$$

This contradicts the choice of $\lambda_\bullet$. Therefore, $\dim C^{\lambda_\bullet}_{\mu_\bullet}(b) = 0$.

It remains to show the uniqueness of $\lambda_\bullet$ with $\dim C^{\lambda_\bullet}_{\mu_\bullet}(b) = 0$. We need two Claims.

By assumption, $b = \dot{\tilde{w}}$ for some $\tilde{w} = u^\tau w \in \tilde{W}$ such that the fixed point $e \in Y_{\mathbb{R}}$ of $\tilde{w}\sigma$ lies in the fundamental alcove Δ. Set $e_\bullet = (e, \dots, e) \in (Y_{\mathbb{R}})^d$ and $\tilde{w}_\bullet = (1, \dots, 1, \tilde{w}) = u^{\tau_\bullet} w_\bullet$ with $\tau_\bullet = (0, \dots, 0, \tau)$ and $w_\bullet = (1, \dots, 1, w)$. Let $\hat{\lambda}_\bullet = \lambda_\bullet - e_\bullet \in (\mathbb{R}^n)^N$.

Claim 1: $\hat{\lambda}^k = \begin{cases} \epsilon^k w^k(\hat{\lambda}^{k+1}), & \text{if } m^k = 0; \\ \varsigma(\epsilon^k w^k(\hat{\lambda}^{k+1})), & \text{if } m^k = 1. \end{cases}$

Claim 2: there exists $1 \le k_0 \le N$ such that $h(\hat{\lambda}^{k_0}) = 0$.

Assume these two Claims for the moment. Suppose there exists another cocharacter $\eta_\bullet \in Y^d$ with $\dim C^{\eta_\bullet}_{\mu_\bullet}(b) = 0$. Let $\hat{\eta}_\bullet = \eta_\bullet - e_\bullet$. Then $\langle \hat{\lambda}^k \rangle = \langle \hat{\eta}^k \rangle$ for any k by Lemma 4.5. We may assume $\delta(\hat{\lambda}^1) \leqslant \delta(\hat{\eta}^1)$, and it follows from Claim 1, Lemma 4.4 (4) and (5) that $\delta(\hat{\lambda}^k) \leqslant \delta(\hat{\eta}^k)$ and $h(\hat{\lambda}^k) \leqslant h(\hat{\eta}^k)$ for any k. By Claim 2, there exists k_0 such that $h(\hat{\eta}^{k_0}) = 0$ and hence $h(\hat{\eta}^{k_0}) = 0 = h(\hat{\lambda}^{k_0})$. By Lemma 4.4 (3), we have $\hat{\lambda}^{k_0} = \hat{\eta}^{k_0}$, which implies by Claim 1 that $\hat{\lambda}^k = \hat{\eta}^k$ for any k as desired. This proved the uniqueness of $\lambda_\bullet$.

Now we prove Claim 1 and 2. For Claim 1, as $e_\bullet = \tilde{w}_\bullet \sigma_\bullet(e_\bullet) = \tau_\bullet + w_\bullet \sigma_\bullet(e_\bullet)$, we have $\lambda^\natural_\bullet = -\hat{\lambda}_\bullet + w_\bullet \sigma_\bullet(\hat{\lambda}_\bullet)$. Therefore we have $(\lambda^\natural)^k = -\hat{\lambda}^k + \epsilon^k w^k(\hat{\lambda}^{k+1})$. As $\mu_\bullet$ is minuscule, it follows from Proposition 2.6 that $\lambda^\natural_\bullet$ is conjugate to $\mu_\bullet$. It's equivalent to say $(\lambda^\natural)^k$ and $\mu^k = m^k \omega_1^\vee$ are conjugate under $\mathfrak{S}_n$. If $m^k = 0$, Claim 1 follows. Now assume $m^k = 1$, then $(\lambda^\natural)^k$ and $\omega_1^\vee$ are conjugate. So there exists $1 \leqslant i_0 \leqslant n$ such that $(\lambda^\natural)^k(i) = 1$ if $i = i_0$ and $(\lambda^\natural)^k(i) = 0$ otherwise. Equivalently,

$$\hat{\lambda}^k(i) = \begin{cases} \epsilon^k w^k(\hat{\lambda}^{k+1})(i) & \text{if } i \neq i_0 \\ \epsilon^k w^k(\hat{\lambda}^{k+1})(i) - 1 & \text{if } i = i_0. \end{cases}$$

Let $x = w^k(\hat{\lambda}^{k+1})(i_0) \in [\hat{\lambda}^{k+1}]$. Notice that $[\hat{\lambda}^k]$ consists of exactly n elements. In order to prove Claim 1, it suffices to show that $x = \max[\hat{\lambda}^{k+1}]$. If it's not the case, then $w^k(\hat{\lambda}^{k+1})(i_1) = \max[\hat{\lambda}^{k+1}] > x$ for some $1 \leqslant i_1 \neq i_0 \leqslant n$. Thus $\hat{\lambda}^k(i_1) = \epsilon^k \max[\hat{\lambda}^{k+1}]$ and $\hat{\lambda}^k(i_0) = \epsilon^k x - 1$. Let $\alpha = \alpha_{i_1, i_0} \in \Phi$. We will show $\alpha \in R(\lambda_\bullet)$ which contradicts the fact that $\dim C^{\lambda_\bullet}_{\mu_\bullet}(b) = |R(\lambda_\bullet)| = 0$ by Proposition 2.6. Indeed, we have

$$\langle \alpha, (\lambda^\natural)^k \rangle = (\lambda^\natural)^k(i_1) - (\lambda^\natural)^k(i_0) = -1,$$

and moreover,

$$\langle \alpha, \lambda^k \rangle = \langle \alpha, \hat{\lambda}^k \rangle + \langle \alpha, e^k \rangle = \epsilon^k \max[\hat{\lambda}^{k+1}] - (\epsilon^k x - 1) + \langle \alpha, e^k \rangle > 1 + \langle \alpha, e^k \rangle.$$

As e^k lies in the fundamental alcove, we have $-1 < \langle \alpha, e^k \rangle < 1$. Moreover $\langle \alpha, \lambda^k \rangle \geqslant 2$ if $\alpha > 0$ and $\langle \alpha, \lambda^k \rangle \geqslant 1$ if $\alpha < 0$. So $(\lambda^k)_\alpha \geqslant 1$ and hence $\alpha \in R(\lambda_\bullet)$. This finishes the proof of Claim 1.

For Claim 2, suppose that $h(\hat{\lambda}^{k+1}) \geqslant 1$ for any k. It follows from Claim 1 that $\min[\hat{\lambda}^k] = \epsilon^k \min[\hat{\lambda}^{k+1}]$ for any k. In particular,

$$\min[\hat{\lambda}^k] = (\prod_{l=1}^{N} \epsilon^l) \min[\hat{\lambda}^k] = p^f \min[\hat{\lambda}^k],$$

which means that $\min[\hat{\lambda}^k] = 0$. This is impossible since b is simple and $[e^k] \cap \mathbb{Z} = \emptyset$ by Proposition 3.3. This proves Claim 2.

It remains to show $C_{\mu_\bullet}(b)$ is connected. For any $\nu_\bullet \in Y^d$ such that $C^{\nu_\bullet}_{\eta_\bullet}(b) \neq \emptyset$, by Lemma 2.6, $C^{\nu_\bullet}_{\mu_\bullet}(b)$ is irreducible. It suffices to show that $C^{\nu_\bullet}_{\mu_\bullet}(b)$ and $C^{\lambda_\bullet}_{\mu_\bullet}(b)$ are in the same geometrically connected component. We argue by induction on $d_{\nu_\bullet} = \dim I^d u^{\nu_\bullet} G^d(\mathcal{O}_L)/G^d(\mathcal{O}_L)$. If $\dim C^{\nu_\bullet}_{\mu_\bullet}(b) = 0$, then $\nu_\bullet = \lambda_\bullet$ and the statement follows. Otherwise, $\overline{C^{\nu_\bullet}_{\mu_\bullet}(b)}$ intersects $I^d u^{\chi_\bullet} G^d(\mathcal{O}_L)/G^d(\mathcal{O}_L)$ for some $\chi_\bullet$ such that $d_{\chi_\bullet} < d_{\nu_\bullet}$. By induction hypothesis, $C^{\chi_\bullet}_{\mu_\bullet}(b)$ is connected to $C^{\lambda_\bullet}_{\mu_\bullet}(b)$. So $C^{\nu_\bullet}_{\mu_\bullet}(b)$ and $C^{\lambda_\bullet}_{\mu_\bullet}(b)$ are in the same geometrically connected component as desired. □

4.2. Now we show the second connectedness result for the totally ramified case with $n = 3$ (i.e. $G = \mathrm{GL}_3$).

Theorem 4.6. *Suppose $G = \mathrm{GL}_3$ and b is simple, then $C_\mu(b)$ is geometrically connected.*

Proof. By Proposition 3.2 and 3.3, we may assume $b = au^\tau\omega$ satisfies Lemma 2.2 with $a \in G(\bar{\mathbb{F}}_p)$ central, $\tau \in Y$ and $w \in \mathfrak{S}_3$ of order 3. Obviously $C_\mu(b) = C_\mu(u^\tau\omega)$, so we may assume $a = 1$. Let

$$S := \{\lambda \in Y | \lambda^\natural \leq \mu\}$$

where $\lambda^\natural := -\lambda + \tau + w\sigma(\lambda)$ as before. By Proposition 2.4, it suffices to connect the points $\{u^\lambda \in C_\mu(b)(\bar{\mathbb{F}}_p) | \lambda \in S\}$ inside $C_\mu(b)$. The result follows from the following two claims.

Claim 1: Suppose $\lambda' = \lambda - \alpha^\vee$ with $\lambda, \lambda' \in S$ and $\alpha^\vee$ is a coroot, then u^λ and $u^{\lambda'}$ are in the same geometrically connected component of $C_\mu(b)$.

Claim 2: For any different $\lambda, \lambda' \in S$, there exists a chain of elements $\lambda_0, \cdots \lambda_r \in S$ for some $r \in \mathbb{N}$ such that $\lambda = \lambda_0$, $\lambda' = \lambda_r$ and $\lambda_{i+1} - \lambda_i$ is a coroot for any $0 \leq i \leq r-1$.

We first prove Claim 1. Let $U_\alpha \subset G$ be the root subgroup of G corresponding to α. We fix an isomorphism $\mathbb{G}_a \simeq U_\alpha$. For any $\bar{\mathbb{F}}_p$-algebra R, we write $U_\alpha(x) \in U_\alpha(R)$ for the image of $x \in R$ via $\mathbb{G}_a \simeq U_\alpha$.

For any $x \in \bar{\mathbb{F}}_p$, let $g(x) := u^\lambda U_\alpha(u^{-1}x)$.

We verify that $g(x)G(\mathcal{O}_L) \in C_\mu(b)(\bar{\mathbb{F}}_p)$. Note that

$$g(x)^{-1} b\sigma g(x) = U_\alpha(-u^{-1}x) u^{\lambda^\natural} U_{w\alpha}(u^{-p}x).$$

By a SL_2-computation, we get

$$U_\alpha(u^{-m}x) \in U_{-\alpha}(u^m x^{-1}) u^{-m\alpha^\vee} G(\mathcal{O}_L)$$

for any positive integer m. So $g(x)^{-1} b\sigma g(x) \in \overline{G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L)}$ if

$$\lambda^\natural - pw\alpha^\vee \leq \mu, \lambda^\natural + \alpha^\vee \leq \mu \text{ and } \lambda'^\natural = \lambda^\natural + \alpha^\vee - pw\alpha^\vee \leq \mu.$$

The first two conditions are implied by the third one as $\langle \alpha, w\alpha^\vee \rangle = -1$, and the third condition is automatic as $\lambda' \in S$.

Therefore, $g(x)$ defines an affine line $g : \mathbb{A}^1 \to C_\mu(b)$. As $C_\mu(b)$ is projective, g extends uniquely to $g : \mathbb{P}^1 \to C_\mu(b)$. It's easy to compute $g(\infty) = u^{\lambda'}$. Hence u^λ and $u^{\lambda'}$ are in the same connected component.

Now it remains to prove Claim 2. Let $\lambda, \lambda' \in S$. Then $\lambda - \lambda'$ is in the coroot lattice. Indeed, write $\lambda = (\lambda(1), \lambda(2), \lambda(3))$ and similary for λ', τ and μ. In order to show $\lambda - \lambda'$ is in the coroot lattice, it suffices to show that $\sum_{i=1}^3 \lambda(i) = \sum_{i=1}^3 \lambda'(i)$. Without loss of generality, we may assume $\omega = (123)$ (and the computation for the other cases are the same). By the definition of $\lambda^\natural$, we get

$$\lambda^\natural = (-\lambda(1) + \tau(1) + p\lambda(3), -\lambda(2) + \tau(2) + p\lambda(1), -\lambda(3) + \tau(3) + p\lambda(2)).$$

The condition $\lambda^\natural \leq \mu$ implies that

$$\begin{aligned}&(p-1)\sum_{i=1}^3 \lambda(i) + \sum_{i=1}^3 \tau(i)\\ =&(-\lambda(1) + \tau(1) + p\lambda(3)) + (-\lambda(2) + \tau(2) + p\lambda(1)) + (-\lambda(3) + \tau(3) + p\lambda(2))\\ =&\sum_{i=1}^3 \mu(i).\end{aligned}$$

We have a similar condition for λ'. Then $\sum_{i=1}^3 \lambda(i) = \sum_{i=1}^3 \lambda'(i)$ follows from the difference of the two equalities for λ and λ'.

Note that $\alpha_1^\vee + w\alpha_1^\vee + w^2\alpha_1^\vee = 0$ for any coroot $\alpha_1^\vee$. It follows that we can always find a coroot $\alpha_1^\vee$ such that

$$\lambda' - \lambda = n_1\alpha_1^\vee + n_2 w\alpha_1^\vee$$

with $n_1 = \max\{|n_1|, |n_2|, |n_1 - n_2|\}$. In particular $n_1 \geq n_2 \geq 0$. We will prove by induction on n_1. Let $\alpha_i^\vee := w^{i-1}\alpha_1^\vee$ for $i \in \mathbb{N}$.

If $n_2 = n_1$ or 0, then $\lambda' - \lambda = n_1\alpha^\vee$ is a multiple of the coroot $\alpha^\vee$ where $\alpha_1^\vee$ equals to $-\alpha_3^\vee$ (resp. $\alpha_1^\vee$) if $n_2 = n_1$ (resp. $n_2 = 0$). Then $\lambda + n\alpha^\vee \in S$ for $0 \leq n \leq n_1$. We are done.

Now we may assume $n_1 > n_2 > 0$.

Claim 3: $\lambda + \alpha_1^\vee \in S$ or $\lambda - \alpha_3^\vee \in S$.

Suppose Claim 3 holds for the moment. Notice that $-\alpha_3^\vee = \alpha_1^\vee + \alpha_2^\vee$, we can apply induction hypothesis to the pair $(\lambda - \alpha_3^\vee, \lambda')$ if $\lambda - \alpha_3^\vee \in S$ or to the pair $(\lambda + \alpha_1^\vee, \lambda')$ if $\lambda + \alpha_1^\vee \in S$. This proves Claim 1.

Now it remains to prove Claim 3. Suppose Claim 3 does not hold. Without loose of generality, we may assume $\alpha_1 = (1, -1, 0)$ and $\alpha_2 = (0, 1, -1)$. Then

$$\begin{aligned}\lambda'^\natural &= \lambda^\natural - (n_1 + pn_2)\alpha_1^\vee + (pn_1 - (p+1)n_2)\alpha_2^\vee\\ &= \lambda^\natural + (-n_1 - pn_2, (p+1)n_1 - n_2, -pn_1 + (p+1)n_2).\end{aligned}$$

In particular,

$$\begin{aligned}(\lambda + \alpha_1^\vee)^\natural &= \lambda^\natural + (-1, p+1, -p)\\ (\lambda - \alpha_3^\vee)^\natural &= \lambda^\natural + (-1-p, p, 1).\end{aligned}$$

Write $\lambda^\natural = (a_1, a_2, a_3)$. As $\lambda'^\natural \leq \mu$ and $\lambda^\natural \leq \mu$, we deduce that $(\lambda + \alpha_1^\vee)^\natural \nleq \mu$ is equivalent to $a_3 - p < \min[\mu]$ and $(\lambda - \alpha_3^\vee)^\natural \nleq \mu$ is equivalent to $a_3 + 1 > \max[\mu]$, where $\max[\mu] = \max\{\mu(1), \mu(2), \mu(3)\}$ and $\min[\mu] = \min\{\mu(1), \mu(2), \mu(3)\}$. Therefore,

$$\max[\mu] - \min[\mu] < 1 + p.$$

On the other hand, the fact that $\lambda'^\natural \leq \mu$ and $\lambda^\natural \leq \mu$ implies

$$\max[\mu] - \min[\mu] \geq n_1 + pn_2 \geq p + 1,$$

which is impossible. □

4.3. In general, $C_\mu(b)$ is not connected. We give two counter-examples.

Proposition 4.7. *(a) Let $G = \mathrm{GL}_4$, $b = u^{(2,0,2,0)}(1243) \in G(\mathbb{F}_p((u)))$ and $\mu = (2p-1, p, p, 1)$. Then*

$$C_\mu(b)(\mathbb{F}_p) = C_\mu(b)(\bar{\mathbb{F}}_p) = \{u^{(2,1,1,0)}, u^{(1,1,1,1)}\}.$$

(b) Let $G = \mathrm{Res}_{k|\mathbb{F}_p}\mathrm{GL}_3$ with $[k : \mathbb{F}_p] = 2$. Choose $\mathbb{F}$ containing k. Then the group $G_{|\mathbb{F}} \simeq \mathrm{GL}_3 \times \mathrm{GL}_3$. Let $b = (u^{(2,0,1)}(123), u^{(0,0,1)}) \in G(\mathbb{F}((u)))$ and $\mu = ((p+1, 0, 0), (p, p, 0))$, then

$$C_\mu(b)(\mathbb{F}) = C_\mu(b)(\bar{\mathbb{F}}_p) = \{u^\chi, u^{\chi'}\},$$

where $\chi = ((1, 0, 1), (0, 0, 1))$ and $\chi' = ((1, 1, 0), (1, 0, 0))$.

Proof. For $\lambda \in Y$ we set $D(\lambda) = \{\alpha \in \Phi; \lambda_\alpha \geqslant 0, \langle \alpha, \lambda^\natural \rangle \leqslant -1\}$.

Claim: if $\lambda^\natural$ is conjugate to μ, and $\lambda_\alpha = 0$ for $\alpha \in D(\lambda)$, then $C_\mu^\lambda(b) = \{u^\lambda\}$.

Indeed, as $\lambda^\natural$ is conjugate to μ, we have

$$U_\lambda^+ u^{\lambda^\natural} \cap \overline{G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L)} = U_\lambda^+(\mathcal{O}_L) u^{\lambda^\natural} U_\lambda^+(\mathcal{O}_L) = U_\lambda^+(\mathcal{O}_L)(\prod_{\alpha \in D(\lambda)} U_\alpha(u^{\langle \alpha, \lambda^\natural \rangle}\mathcal{O}_L)) u^{\lambda^\natural}.$$

For $\alpha \in D$ we have $\lambda_\alpha = 0$ by assumption and hence

$$I \cap U_\alpha(u^{\langle \alpha, \lambda \rangle + \langle \alpha, \lambda^\natural \rangle}\mathcal{O}_L) = u^\lambda U_\alpha(\mathcal{O}_L) u^{-\lambda}.$$

Thus

$$\begin{aligned} &(I \cap U_\lambda^+) \cap u^\lambda \overline{G(\mathcal{O}_L) u^\mu G(\mathcal{O}_L)} u^{-\lambda^\dagger} \\ &= u^\lambda U_\lambda^+(\mathcal{O}_L) u^{-\lambda} \prod_{\alpha \in D(\lambda)} I \cap U_\alpha(u^{\langle \alpha, \lambda \rangle + \langle \alpha, \lambda^\natural \rangle}\mathcal{O}_L) \\ &= u^\lambda U_\lambda^+(\mathcal{O}_L) u^{-\lambda}, \end{aligned}$$

which means $C_\mu^\lambda(b) = \{u^\lambda\}$ by Proposition 2.4 (b). This concludes the Claim.

In the case (a), one checks directly that $C_\mu^\lambda(b) \neq \emptyset$, or equivalently $\lambda^\natural \leq \mu$, if and only if $\lambda = (1,1,1,1)$ or $\lambda = (2,1,1,0)$. In the former case, we have $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L) = u^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ and $C_\mu^\lambda = \{u^\lambda\}$ since λ is central. In the latter case, one checks that λ satisfies the conditions of the Claim and it follows that $C_\mu^\lambda(b) = \{u^\lambda\}$ as desired.

In the case (b), one checks directly that $C_\mu^\lambda(b) \neq \emptyset$ if and only if $\lambda = \chi'$ or $\lambda = \chi$. If $\lambda = \chi'$, then λ is dominant and minuscule, which means $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L) = u^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ and $C_\mu^\lambda(b) = \{u^\lambda\}$. If $\lambda = \chi$, then the conditions of the Claim are satisfied and hence $C_\mu^\lambda(b) = \{u^\lambda\}$ as desired. □

5. Application to deformation spaces

In this section, we assume $p > 2$. Suppose $\bar{\rho} : \Gamma_K \to \mathrm{GL}_n(\mathbb{F})$ is absolutely irreducible and flat. Here flat means that $\bar{\rho}$ comes from a finite flat group scheme over $\mathcal{O}_K$. Let R^{fl} be the flat deformation ring of $\bar{\rho}$ in the sense of Ramakrishna ([45]). It's a complete local noetherian $W(\mathbb{F})$-algebra.

We consider the conjugacy class $\{\nu\}$ of a minuscule cocharacter

$$\nu : \mathbb{G}_{m,\bar{\mathbb{Q}}_p} \to (\mathrm{Res}_{K|\mathbb{Q}_p}\mathrm{GL}_n)_{\bar{\mathbb{Q}}_p}.$$

Let T_n be the maximal torus of GL_n consisting of diagonal matrices. We may assume that ν is dominant and has image in $\mathrm{Res}_{K|\mathbb{Q}_p}T_{n,\bar{\mathbb{Q}}_p}$. Write $\nu = (\nu_\delta)_{\delta\in\mathrm{Hom}(K,\bar{\mathbb{Q}}_p)}$ with

$$\nu_\delta = (\underbrace{1,\cdots,1}_{v_\delta},0,\cdots,0) \in X_*(T_n) = \mathbb{Z}^n.$$

We deonote by $R^{fl,\nu}$ for the quotient of R^{fl} corresponding to the deformations $\xi : G_K \to \mathrm{GL}_n(\mathcal{O}_E)$ with E an extension of $W(\mathbb{F})[\frac{1}{p}]$ that contains the reflex field of $\{\nu\}$, such that Hodge-Tate weights given by ν, i.e., for any $a \in K$,

$$\det_E(a|D_{cris}(\xi)_K/\mathrm{Fil}^0 D_{cris}(\xi)_K) = \prod_{\delta\in\mathrm{Hom}(K,\bar{\mathbb{Q}}_p)} \delta(a)^{v_\delta}.$$

Corollary 5.1. *The scheme* $\mathrm{Spec}(R^{\mathrm{fl},\nu}[\frac{1}{p}])$ *is connected if one of the following two conditions holds:*

(1) $\nu_\tau = (1,0\cdots,0)$ *or central for all* $\tau \in \mathrm{Hom}(K,\bar{\mathbb{Q}}_p)$*;*
(2) K *is totally ramified and* $n = 3$*.*

Proof. Suppose the restriction to G_{K_∞} of the Tate-Twist $\bar{\rho}(-1)$ corresponding to an absolutely simple φ-module

$$((k\otimes_{\mathbb{F}_p}\mathbb{F})^n, b\varphi)$$

of rank n over $k\otimes_{\mathbb{F}_p}\mathbb{F}$, where $b \in \mathrm{Res}_{k|\mathbb{F}_p}\mathrm{GL}_n(\mathbb{F})$.

The cocharcter ν comes from a cocharacter over $\bar{\mathbb{Z}}_p$ that we still denote by ν:

$$\nu : \mathbb{G}_{m,\bar{\mathbb{Z}}_p} \to (\mathrm{Res}_{K|\mathbb{Z}_p}T_n)_{\bar{\mathbb{Z}}_p}.$$

Therefore, we obtain a cocharacter

$$\mu(\nu) : \mathbb{G}_{m,\bar{\mathbb{F}}_p} \xrightarrow{\nu\times_{\bar{\mathbb{Z}}_p}\bar{\mathbb{F}}_p} (\mathrm{Res}_{\mathcal{O}_K|\mathbb{Z}_p}(T_n))\times_{\bar{\mathbb{Z}}_p}\bar{\mathbb{F}}_p \to \mathrm{Res}_{k|\mathbb{F}_p}(T_n)_{\bar{\mathbb{F}}_p},$$

where the second arrow is the natural map induced from $\mathcal{O}_K\otimes_{\mathbb{Z}_p}\bar{\mathbb{F}}_p \to k\otimes_{\mathbb{F}_p}\bar{\mathbb{F}}_p$. More concretely, write $\mu(\nu) = (\mu(\nu)_\tau)_{\tau\in\mathrm{Hom}(k,\bar{\mathbb{F}}_p)}$ with $\mu(\nu)_\tau \in X_*(T_n) = \mathbb{Z}^n$. Then for any $\tau \in \mathrm{Hom}(k,\bar{\mathbb{F}}_p)$,

$$\mu(\nu)_\tau = \sum_{\substack{\delta\in\mathrm{Hom}(K,\bar{\mathbb{Q}}_p)\\ \bar{\delta}=\tau}} \nu_\delta,$$

where $\bar{\delta}$ is the embedding of the residue fields induced from δ.

By [31, 2.4.10], the connected components of $\mathrm{Spec}(R^{\mathrm{fl},\nu}[\frac{1}{p}])$ is in bijection with $C_{\mu(\nu)}(b)$ (We use different notations as in loc. cit., the deformation ring $R^{\mathrm{fl},\nu}$ is denoted by R^{v} in loc. cit. and the Kisin variety $C_{\mu(\nu)}(b)$ is denoted by

$\mathcal{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$ in loc. cit.). Note that condition (1) is equivalent to $\mu(\nu) = (\mu(\nu)_\tau)_\tau$ with $\mu(\nu)_\tau = (\mu(\nu)_\tau(1), \cdots, \mu(\nu)_\tau(n))$ such that $\mu(\nu)_\tau(1) \geq \mu(\nu)_\tau(2) = \cdots = \mu(\nu)_\tau(n)$ for all $\tau \in \mathrm{Hom}(K, \bar{\mathbb{Q}}_p)$ which is exactly the condition of μ in Theorem 4.1. Hence the result follows from Theorems 4.1 and 4.6. □

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